

# Proposal of a Simple Method to Estimate Neutrino Oscillation Probability and CP Violation in Matter

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## Abstract

We study neutrino oscillation within the framework of three generations in matter. We propose a simple method to approximate the coefficients  $A$ ,  $B$  and  $C$  which do not depend on the CP phase  $\delta$  in the oscillation probability  $P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C$ . An advantage of our method is that an approximate formula of the coefficients  $A$ ,  $B$  and  $C$  in arbitrary matter *without the usual first order perturbative calculations* of the small parameter  $\Delta m_{21}^2/\Delta m_{31}^2$  or  $\sin \theta_{13}$  can be derived. Furthermore we show that all the approximate formulas for low, intermediate and high energy regions given by other authors in constant matter can be easily derived from our formula. It means that our formula is applicable over a wide energy region.

## 1 Introduction

Recent experiments clarified that the solar neutrino deficit and the atmospheric neutrino anomaly are strong evidences for the neutrino oscillations with three generations. The solar neutrino deficit is explained by  $\nu_e \rightarrow \nu_\mu$  oscillation [1] and the atmospheric neutrino anomaly is explained by  $\nu_\mu \rightarrow \nu_\tau$  oscillation [2]. In the recent SNO [3] and KamLAND experiments [4], the solar neutrino problem has been solved by the large mixing angle (LMA) MSW solution [5]. Furthermore the upper bound of  $\theta_{13}$  is given by the CHOOZ experiment [6]. Thus, there are two small parameters

$$\alpha \equiv \Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03, \quad \sin \theta_{13} \leq 0.16. \quad (1)$$

The remaining problems are the determination of sign  $\Delta m_{31}^2$ , the measurement of the 1-3 mixing angle  $\theta_{13}$  and the CP phase  $\delta$  [7]. In the limit of vanishing mixing angle  $\theta_{13}$  or vanishing mass squared difference  $\Delta m_{21}^2$ , the CP violating effects in the oscillation probability disappear. Therefore the magnitude of the two small parameters  $\alpha$  and  $\sin \theta_{13}$  controls the magnitude of the CP violation. The LMA MSW solution in the solar neutrino problem has opened the possibility of the observation of CP violation in the lepton sector. For this purpose, many long baseline neutrino experiments are planned [8].

The matter effect received from the earth is important in the long baseline neutrino experiments, because fake CP violation is induced due to matter effect. The Preliminary Reference Earth Model (PREM) is well known as the model of the earth density and is usually used in analysis of long baseline experiments.

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However, it has recently been pointed out in geophysical analysis of the matter density profile from J-PARC to Beijing [9] that the deviation from the PREM is rather large. In this paper, we derive an approximate formula of neutrino oscillation probability without assuming any specific earth density models.

In constant matter, various approximate formulas have been proposed in low energy [10, 11, 12], in intermediate energy [13, 14, 15] and in high energy regions [16, 17, 18]. In the case that the matter density is not constant, approximate formulas have been also derived in [19, 20, 21, 22, 23, 24, 25] using perturbative calculations to analyze the terrestrial matter effect. However, the question of how to separate the genuine CP violation due to the leptonic CP phase from the fake CP violation induced by matter effect has not been investigated sufficiently in arbitrary matter.

The next step is to analyze the CP violating effects in more detail in the case of non-constant matter density. In order to obtain a hint for this problem, we will briefly review the approach applied in the solar neutrino problem. It is difficult to derive the exact solutions for solar neutrino problem in three generations except for some special matter profile. As an approach to derive the neutrino oscillation probability, a low energy approximate formula was proposed in [26]. By averaging  $\Delta m_{31}^2$ , they derived the formula

$$P^{(3)}(\nu_e \rightarrow \nu_e) = \cos^4 \theta_{13} P^{(2)}(\nu_e \rightarrow \nu_e) + \sin^4 \theta_{13}. \quad (2)$$

This is a formula to reduce the calculation of the survival probability  $P^{(3)}(\nu_e \rightarrow \nu_e)$  in three generations to that of  $P^{(2)}(\nu_e \rightarrow \nu_e)$  in two generations. Therefore this formula is called the reduction formula [27]. This reduction formula is useful for the analysis of solar neutrino experiments, but it is not directly applicable to long baseline neutrino experiments planned in the future, because we cannot average  $\Delta m_{31}^2$  in long baseline experiments. Therefore we need to derive the reduction formula which is valid without averaging  $\Delta m_{31}^2$ .

In a series of previous papers we have calculated the oscillation probability  $P(\nu_e \rightarrow \nu_\mu)$ . In the papers [28] we have shown that the CP phase  $\delta$  dependence of  $P(\nu_e \rightarrow \nu_\mu)$  in constant matter is given in the form

$$P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C \quad (3)$$

and have derived an exact but simple expression for the coefficients  $A$ ,  $B$  and  $C$ . In the next paper [29] we have presented a simple and general formula which does not depend on the matter profile. As a result we have concluded that the equation (3) is valid even in arbitrary matter. However, in the case of non-constant matter density, there exist no closed-form expressions for the coefficients  $A$ ,  $B$  and  $C$ .

In this paper we propose a simple method to derive the approximate formula of the coefficients  $A$ ,  $B$  and  $C$  taking account of the small parameters  $\alpha$  and  $\sin \theta_{13}$ . The coefficients  $A$  and  $B$  are linear in  $\alpha$  and  $\sin \theta_{13}$ . These coefficients represent the genuine three flavor effect. Therefore it has been considered that the first order perturbative calculations of  $\alpha$  or  $\theta_{13}$  are needed for the derivation of  $A$  and  $B$ . However it is possible to calculate  $A$ ,  $B$  and  $C$  without the usual first order perturbative calculations of small parameter  $\alpha$  or  $\sin \theta_{13}$  in our method. As we shall see later in section 2, the reduction formula in arbitrary matter is derived as

$$A \simeq 2\text{Re}[S_{\mu e}^{\ell*} S_{\tau e}^h] c_{23} s_{23}, \quad (4)$$

$$B \simeq -2\text{Im}[S_{\mu e}^{\ell*} S_{\tau e}^h] c_{23} s_{23}, \quad (5)$$

$$C \simeq |S_{\mu e}^\ell|^2 c_{23}^2 + |S_{\tau e}^h|^2 s_{23}^2, \quad (6)$$

where  $S_{\mu e}^\ell$  and  $S_{\tau e}^h$  are the oscillation amplitudes calculated in the following Hamiltonian, respectively

$$H^\ell = O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T + \text{diag}(a(t), 0, 0), \quad (7)$$

$$H^h = O_{13} \text{diag}(0, 0, \Delta_{31}) O_{13}^T + \text{diag}(a(t), 0, 0). \quad (8)$$

Here  $\Delta_{ij}$  is defined by  $\Delta_{ij} = \Delta m_{ij}^2 / 2E$ ,  $O_{ij}$  is the rotational matrix in the  $ij$  plane and  $a(t)$  is the matter potential. Since both  $H^\ell$  and  $H^h$  are Hamiltonians in two generations, the equations (4)-(6) are formulas

in order to reduce the calculation of the coefficients  $A, B$  and  $C$  in three generations to the oscillation amplitudes in two generations. Furthermore we show that other approximate formulas known in constant matter can be easily derived from our formula. This means that our formula is applicable to a wide energy region.

## 2 New Idea for an Approximate Formula

In this section we propose a new idea to derive an approximate formula for neutrino oscillation probability. At first we review a general framework for the oscillation probability in arbitrary matter. Next we introduce how to derive an approximate formula from this framework. We also discuss the difference between our method and usual methods.

### 2.1 Review of General Formulation

In this subsection we briefly review that the CP dependence of the oscillation probability  $P(\nu_e \rightarrow \nu_\mu)$  is given in the form as  $P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C$  in arbitrary matter. More detailed calculation has been given in the papers [29].

The Hamiltonian in matter is given by

$$H = U \text{diag}(0, \Delta_{21}, \Delta_{31}) U^\dagger + \text{diag}(a(t), 0, 0), \quad (9)$$

where  $U$  is the Maki-Nakagawa-Sakata (MNS) matrix [30]. Using the standard parametrization

$$U = O_{23} \Gamma O_{13} \Gamma^\dagger O_{12}, \quad (10)$$

the Hamiltonian is written as

$$H = O_{23} \Gamma O_{13} O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T \Gamma^\dagger O_{23}^T + \text{diag}(a(t), 0, 0), \quad (11)$$

where  $\Gamma = \text{diag}(1, 1, e^{i\delta})$  is the phase matrix, It should be pointed out that the matter potential of the Hamiltonian contains only the  $e$ - $e$  component. The Hamiltonian can be written in the form decomposing the CP phase  $\delta$  and the 2-3 mixing angle  $\theta_{23}$  as

$$H = O_{23} \Gamma H' \Gamma^\dagger O_{23}^T, \quad (12)$$

where  $H'$  is defined as

$$H' = O_{13} O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T + \text{diag}(a(t), 0, 0). \quad (13)$$

The amplitudes  $S$  and  $S'$  are given by substituting the relations (12) and (13) into the following equations

$$S = T \exp \left\{ -i \int_0^L H(t) dt \right\}, \quad S' = T \exp \left\{ -i \int_0^L H'(t) dt \right\}. \quad (14)$$

Then we obtain the amplitude for  $\nu'_\beta \rightarrow \nu'_\alpha$  as the  $\alpha$ - $\beta$  component

$$S_{\alpha\beta} = (O_{23} \Gamma S' \Gamma^\dagger O_{23}^T)_{\alpha\beta}. \quad (15)$$

In particular when we choose  $\mu$  and  $e$  as  $\alpha$  and  $\beta$ , the amplitude  $S_{\mu e}$  is given by

$$S_{\mu e} = S'_{\mu e} c_{23} + S'_{\tau e} s_{23} e^{i\delta}. \quad (16)$$

From this relation, the probability is calculated as

$$P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C, \quad (17)$$

$$A = 2\text{Re}[S'_{\mu e}^* S'_{\tau e}] c_{23} s_{23}, \quad (18)$$

$$B = -2\text{Im}[S'_{\mu e}^* S'_{\tau e}] c_{23} s_{23}, \quad (19)$$

$$C = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}|^2 s_{23}^2, \quad (20)$$

which is the exact formula in arbitrary matter derived in the previous paper [29].

## 2.2 Order Counting of $A, B$ and $C$ on $\alpha$ and $\sin \theta_{13}$

In this subsection we study how the coefficients  $A, B$  and  $C$  defined in (18)-(20) depend on  $\alpha$  and  $\sin \theta_{13}$ . Instead of  $A, B$  and  $C$ , we study the dependence of  $S'_{\mu e}$  and  $S'_{\tau e}$  on  $\alpha$  and  $\sin \theta_{13}$  by taking the limit either  $\theta_{13} \rightarrow 0$  or  $\alpha \rightarrow 0$ .

At first, taking the limit  $\theta_{13} \rightarrow 0$ , the Hamiltonian reduces to

$$H^\ell = \lim_{\theta_{13} \rightarrow 0} H' \quad (21)$$

$$= O_{12}\text{diag}(0, \Delta_{21}, \Delta_{31})O_{12}^T + \text{diag}(a(t), 0, 0) \quad (22)$$

$$= \begin{pmatrix} \Delta_{21}s_{12}^2 + a(t) & \Delta_{21}s_{12}c_{12} & 0 \\ \Delta_{21}s_{12}c_{12} & \Delta_{21}c_{12}^2 & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix}. \quad (23)$$

This Hamiltonian expresses the fact that the third generation is separated from the first and the second generations. We simply obtain the amplitude

$$S'_{\tau e} = 0 \quad (24)$$

from the Hamiltonian (23). It means that the order of  $S'_{\tau e}$  is given by

$$S'_{\tau e} = O(\sin \theta_{13}). \quad (25)$$

for the case  $\theta_{13} \neq 0$ . In the same way, taking the limit  $\Delta_{21} \rightarrow 0$ , the Hamiltonian reduces to

$$H^h = \lim_{\Delta_{21} \rightarrow 0} H' \quad (26)$$

$$= O_{13}\text{diag}(0, 0, \Delta_{31})O_{13}^T + \text{diag}(a(t), 0, 0) \quad (27)$$

$$= \begin{pmatrix} \Delta_{31}s_{13}^2 + a(t) & 0 & \Delta_{31}s_{13}c_{13} \\ 0 & 0 & 0 \\ \Delta_{31}s_{13}c_{13} & 0 & \Delta_{31}c_{13}^2 \end{pmatrix}. \quad (28)$$

This Hamiltonian expresses the fact that the second generation is separated from the first and the third generations. We simply obtain the amplitude

$$S'_{\mu e} = 0 \quad (29)$$

from the Hamiltonian (28). It means that the order of  $S'_{\mu e}$  is given by

$$S'_{\mu e} = O(\alpha) \quad (30)$$

for the case  $\alpha \neq 0$ . Finally, we conclude that the dependence of the coefficients  $A, B$  and  $C$  on  $\alpha$  and  $\sin \theta_{13}$  is given by

$$A = 2\text{Re}[S'_{\mu e} S'_{\tau e}] c_{23} s_{23} = O(\alpha \sin \theta_{13}), \quad (31)$$

$$B = -2\text{Im}[S'_{\mu e}^* S'_{\tau e}] c_{23} s_{23} = O(\alpha \sin \theta_{13}), \quad (32)$$

$$C = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}|^2 s_{23}^2 = O(\alpha^2) + O(\sin^2 \theta_{13}). \quad (33)$$

Since both  $A$  and  $B$  vanish in the two flavor limit, either  $\alpha \rightarrow 0$  or  $\sin \theta_{13} \rightarrow 0$ , this fact represents the genuine three flavor effect. The coefficients  $A$  and  $B$  are doubly suppressed by these small parameters  $\alpha$  and  $\sin \theta_{13}$ . In Refs. [17, 18], this is pointed out for the case of constant matter density. However these results (31)-(33) are correct even in arbitrary matter profile.

### 2.3 Main Result

In this subsection, we propose a simple method to approximately calculate the amplitudes  $S'_{\mu e}$  and  $S'_{\tau e}$ . From the result of the previous subsection, the dependence of  $S'_{\mu e}$  and  $S'_{\tau e}$  on  $\alpha$  and  $\sin \theta_{13}$  is given by

$$S'_{\mu e} = O(\alpha), \quad S'_{\tau e} = O(\sin \theta_{13}). \quad (34)$$

We expand both  $S'_{\mu e}$  and  $S'_{\tau e}$  in terms of two small parameters  $\alpha$  and  $\sin \theta_{13}$  as

$$S'_{\mu e} = \left( O(\alpha) + O(\alpha^2) + O(\alpha^3) + \dots \right) + \left( O(\alpha \sin \theta_{13}) + O(\alpha^2 \sin \theta_{13}) + \dots \right) \quad (35)$$

$$= S_{\mu e}^\ell + O(\alpha \sin \theta_{13}) + O(\alpha^2 \sin \theta_{13}) + \dots, \quad (36)$$

$$S'_{\tau e} = \left( O(\sin \theta_{13}) + O(\sin^2 \theta_{13}) + \dots \right) + \left( O(\alpha \sin \theta_{13}) + O(\alpha^2 \sin \theta_{13}) + \dots \right) \quad (37)$$

$$= S_{\tau e}^h + O(\alpha \sin \theta_{13}) + O(\alpha^2 \sin \theta_{13}) + \dots, \quad (38)$$

where  $S_{\mu e}^\ell$  and  $S_{\tau e}^h$  are defined by

$$S_{\mu e}^\ell = \lim_{\theta_{13} \rightarrow 0} S'_{\mu e}, \quad (39)$$

$$S_{\tau e}^h = \lim_{\alpha \rightarrow 0} S'_{\tau e}. \quad (40)$$

From (36) and (38) we can approximate the amplitudes as

$$S'_{\mu e} \simeq S_{\mu e}^\ell, \quad (41)$$

$$S'_{\tau e} \simeq S_{\tau e}^h. \quad (42)$$

The accuracy of this approximation is determined by the magnitude of the higher order terms on  $\sin \theta_{13}$  and  $\alpha$ . At present, the upper bound of  $\sin \theta_{13}$  is given by the CHOOZ experiment. In future experiments, when the value of  $\theta_{13}$  will become smaller, the accuracy of the approximate formula can be better. It is noted that the simple method introduced in this subsection does not depend on whether the matter density is constant or not. We obtain the oscillation probability from the reduced amplitudes as

$$P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C, \quad (43)$$

$$A \simeq 2\text{Re}[S_{\mu e}^\ell S_{\tau e}^h] c_{23} s_{23}, \quad (44)$$

$$B \simeq -2\text{Im}[S_{\mu e}^\ell S_{\tau e}^h] c_{23} s_{23}, \quad (45)$$

$$C \simeq |S_{\mu e}^\ell|^2 c_{23}^2 + |S_{\tau e}^h|^2 s_{23}^2. \quad (46)$$

This formula is one of the main results obtained in this paper. The advantage of this formula is as follows. First, this formula is derived by using only two small parameters  $\alpha$  and  $\sin \theta_{13}$  without assuming a specific matter density model. Therefore, this formula is applicable to the case of the PREM, ak135f and so on. Second, the reduction formula (2) applied to the solar neutrino problem is valid only in the case that the averaging for  $\Delta m_{31}^2$  is possible. However, our formula is effective even in the case that the oscillation probability can not be averaged. Namely, it is applicable to long baseline experiments. Third, we derive this formula *without the first order perturbative calculations* of small parameter  $\alpha$  or  $\sin \theta_{13}$ . This is the reason why our derivation is easier than usual perturbative methods proposed by other authors. More detailed discussion is given in the next subsection.

## 2.4 Comparison with Usual Perturbative Calculations

In this subsection, we compare our method with usual perturbative methods and describe the advantage of our method clearly. From the result of the previous subsection, the dependence of the coefficients  $A$  and  $B$  on  $\alpha$  and  $\sin \theta_{13}$  is given by

$$A = O(\alpha \sin \theta_{13}), \quad B = O(\alpha \sin \theta_{13}). \quad (47)$$

As both  $\alpha$  and  $\sin \theta_{13}$  are small parameters, there are two kinds of perturbative methods. One method is to consider  $\alpha$  as a small parameter and treat  $\theta_{13}$  exactly. Another method is to consider  $\sin \theta_{13}$  as a small parameter and treat  $\alpha$  exactly. The former case means that we consider  $H'$  as a perturbation from  $H^\ell$

$$H' = H^\ell + O(\sin \theta_{13}). \quad (48)$$

We need to perform the first order perturbative calculation to obtain  $A$  and  $B$  in this perturbative method. Similarly, the later case means that we consider  $H'$  as a perturbation from  $H^h$

$$H' = H^h + O(\alpha). \quad (49)$$

In order to calculate  $A$  and  $B$ , we need to perform the first order perturbative calculation. In both cases, we need to perform the first order perturbative calculation, because the CP violating effects disappear in the limit of vanishing  $\alpha$  or  $\theta_{13}$ .

Let us interpret the above usual perturbative method by using the general formulation (18) and (19) as follows. The expressions  $A$  and  $B$  are represented by two kinds of amplitudes  $S'_{\mu e}$  and  $S'_{\tau e}$ . The dependence (25) and (30) of the two amplitudes on  $\alpha$  and  $\sin \theta_{13}$  is rewritten as

$$S'_{\mu e} = O(\alpha \sin^0 \theta_{13}), \quad S'_{\tau e} = O(\alpha^0 \sin \theta_{13}). \quad (50)$$

If we use the expansion in terms of  $\sin \theta_{13}$  to calculate both amplitudes  $S'_{\mu e}$  and  $S'_{\tau e}$ , the amplitude  $S'_{\mu e}$  can be calculated in the zeroth order perturbation. However,  $S'_{\tau e}$  need to be calculated in the first order perturbation. In the same way, if we use the expansion in terms of  $\alpha$ , the amplitude  $S'_{\tau e}$  can be calculated in the zeroth order perturbation, but  $S'_{\mu e}$  need to be calculated in the first order perturbation.

An advantage of our method is that we are able to calculate both the amplitudes  $S'_{\mu e}$  and  $S'_{\tau e}$  in the zeroth order perturbation, namely *without the first order perturbation* of the small parameter  $\alpha$  or  $\sin \theta_{13}$ . If we expand  $S'_{\mu e}$  in terms of  $\sin \theta_{13}$  instead of  $\alpha$ , we do not need to perform the first order perturbation. Similarly, if we expand  $S'_{\tau e}$  in terms of  $\alpha$  instead of  $\sin \theta_{13}$ , we do not need to perform the first order perturbation to calculate  $S'_{\tau e}$ . One of the essential points of our method is that the Hamiltonian to calculate  $S'_{\mu e}^\ell$  is different from that to calculate  $S'_{\tau e}^h$ . Another point is that we only have to calculate  $S'_{\mu e}^\ell$  and  $S'_{\tau e}^h$  by using the Hamiltonian  $H^\ell$  and  $H^h$  in the framework of two generations, respectively. These ideas make the calculations of the probability easy.

### 3 Approximate Formula in Vacuum or in Constant Matter

In this section, we calculate the concrete expressions for  $A, B$  and  $C$  both in vacuum and in constant matter by using the new method. Moreover, we compare the value of these coefficients with exact value by numerical calculation.

#### 3.1 In Vacuum

At first, we calculate  $S_{\mu e}^\ell$  in vacuum, namely in the case of  $a(t) = 0$ .  $S_{\mu e}^\ell$  is

$$S_{\mu e}^\ell = (\exp(-iH^\ell L))_{\mu e} \quad (51)$$

$$= (O_{12}\text{diag}(1, e^{-i\Delta_{21}L}, e^{-i\Delta_{31}L})O_{12}^T)_{\mu e} \quad (52)$$

$$= -i \sin 2\theta_{12} \sin \frac{\Delta_{21}L}{2} \exp\left(-i \frac{\Delta_{21}}{2} L\right) \quad (53)$$

from the Hamiltonian (23). Similarly  $S_{\tau e}^h$  is

$$S_{\tau e}^h = -i \sin 2\theta_{13} \sin \frac{\Delta_{31}L}{2} \exp\left(-i \frac{\Delta_{31}}{2} L\right) \quad (54)$$

from the Hamiltonian (28). We obtain the expressions for  $A, B$  and  $C$

$$A \simeq \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{\Delta_{21}L}{2} \sin \frac{\Delta_{31}L}{2} \cos \frac{\Delta_{32}L}{2}, \quad (55)$$

$$B \simeq \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{\Delta_{21}L}{2} \sin \frac{\Delta_{31}L}{2} \sin \frac{\Delta_{32}L}{2}, \quad (56)$$

$$C \simeq c_{23}^2 \sin^2 2\theta_{12} \sin^2 \frac{\Delta_{21}L}{2} + s_{23}^2 \sin^2 2\theta_{13} \sin^2 \frac{\Delta_{31}L}{2}, \quad (57)$$

by substituting (53) and (54) into (44), (45) and (46).

#### 3.2 In Constant Matter

Next we calculate the amplitudes in constant matter, namely in the case of  $a(t) = a$ . At first, we diagonalize the Hamiltonian (23) in constant matter by the orthogonal matrix  $O_{12}^\ell$  as

$$H^\ell = O_{12}\text{diag}(0, \Delta_{21}, \Delta_{31})O_{12}^T + \text{diag}(a, 0, 0) \quad (58)$$

$$= O_{12}^\ell \text{diag}(\lambda_1^\ell, \lambda_2^\ell, \Delta_{31})(O_{12}^\ell)^T \quad (59)$$

to calculate  $S_{\mu e}^\ell$ . Here  $\lambda_i^\ell (i = 1, 2)$  is the eigenvalue given by

$$\lambda_i^\ell = \frac{1}{2} \left( \Delta_{21} + a \pm \sqrt{(\Delta_{21} \cos 2\theta_{12} - a)^2 + \Delta_{21}^2 \sin^2 2\theta_{12}} \right), \quad (60)$$

and  $\lambda_1^\ell$  and  $\lambda_2^\ell$  correspond to the sign  $-$  and the opposite sign  $+$ , respectively. The effective mixing angle  $\sin 2\theta_{12}^\ell$  is calculated as

$$\sin^2 2\theta_{12}^\ell = \frac{\Delta_{21}^2 \sin^2 2\theta_{12}}{(\Delta_{21} \cos 2\theta_{12} - a)^2 + \Delta_{21}^2 \sin^2 2\theta_{12}}. \quad (61)$$

From (60) and (61) we obtain the relation

$$\frac{\Delta_{21}^\ell}{\Delta_{21}} = \frac{\sin 2\theta_{12}}{\sin 2\theta_{12}^\ell} = \sqrt{\left(\cos 2\theta_{12} - \frac{a}{\Delta_{21}}\right)^2 + \sin^2 2\theta_{12}}. \quad (62)$$

The amplitude  $S_{\mu e}^\ell$  is calculated by using the  $\lambda_i^\ell$  and  $\sin 2\theta_{12}^\ell$  as

$$S_{\mu e}^\ell = (\exp(-iH^\ell L))_{\mu e} \quad (63)$$

$$= \left( O_{12} \text{diag}(e^{-i\lambda_1^\ell L}, e^{-i\lambda_2^\ell L}, e^{-i\Delta_{31} L}) O_{12}^T \right)_{\mu e} \quad (64)$$

$$= -i \sin 2\theta_{12}^\ell \sin \frac{\Delta_{21}^\ell L}{2} \exp\left(-i \frac{\lambda_1^\ell + \lambda_2^\ell}{2} L\right) \quad (65)$$

$$= -i \sin 2\theta_{12}^\ell \sin \frac{\Delta_{21}^\ell L}{2} \exp\left(-i \frac{\Delta_{21} + a}{2} L\right). \quad (66)$$

Next let us calculate  $S_{\tau e}^h$  from the Hamiltonian (28) diagonalized by the orthogonal matrix  $O_{13}^h$

$$H^h = O_{13} \text{diag}(0, 0, \Delta_{31}) O_{13}^T + \text{diag}(a, 0, 0) \quad (67)$$

$$= O_{13}^h \text{diag}(\lambda_1^h, 0, \lambda_3^h) (O_{13}^h)^T. \quad (68)$$

The eigenvalue  $\lambda_i^h (i = 1, 3)$  of this Hamiltonian is given by

$$\lambda_i^h = \frac{1}{2} \left( \Delta_{31} + a \pm \sqrt{(\Delta_{31} \cos 2\theta_{13} - a)^2 + \Delta_{31}^2 \sin^2 2\theta_{13}} \right), \quad (69)$$

where  $\lambda_1^h$  and  $\lambda_3^h$  correspond to the sign  $-$  and the opposite sign  $+$ . Moreover we obtain the effective mixing angle  $\sin 2\theta_{13}^h$  is calculated as

$$\sin^2 2\theta_{13}^h = \frac{\Delta_{31}^2 \sin^2 2\theta_{13}}{(\Delta_{31} \cos 2\theta_{13} - a)^2 + \Delta_{31}^2 \sin^2 2\theta_{13}}. \quad (70)$$

From (69) and (70) we obtain the relation

$$\frac{\Delta_{31}^h}{\Delta_{31}} = \frac{\sin 2\theta_{13}}{\sin 2\theta_{13}^h} = \sqrt{\left(\cos 2\theta_{13} - \frac{a}{\Delta_{31}}\right)^2 + \sin^2 2\theta_{13}}. \quad (71)$$

The amplitude  $S_{\tau e}^h$  is calculated as

$$S_{\tau e}^h = -i \sin 2\theta_{13}^h \sin \frac{\Delta_{31}^h L}{2} \exp\left(-i \frac{\Delta_{31} + a}{2} L\right). \quad (72)$$

Substituting (66) and (72) into (44), (45) and (46), we obtain

$$A \simeq \sin 2\theta_{12}^\ell \sin 2\theta_{23} \sin 2\theta_{13}^h \sin \frac{\Delta_{21}^\ell L}{2} \sin \frac{\Delta_{31}^h L}{2} \cos \frac{\Delta_{32} L}{2}, \quad (73)$$

$$B \simeq \sin 2\theta_{12}^\ell \sin 2\theta_{23} \sin 2\theta_{13}^h \sin \frac{\Delta_{21}^\ell L}{2} \sin \frac{\Delta_{31}^h L}{2} \sin \frac{\Delta_{32} L}{2}, \quad (74)$$

$$C \simeq c_{23}^2 \sin^2 2\theta_{12}^\ell \sin^2 \frac{\Delta_{21}^\ell L}{2} + s_{23}^2 \sin^2 2\theta_{13}^h \sin^2 \frac{\Delta_{31}^h L}{2}. \quad (75)$$

The low and high energy MSW effects are contained in  $\sin 2\theta_{12}^\ell, \Delta_{21}^\ell$  and in  $\sin 2\theta_{13}^h, \Delta_{31}^h$  of the approximate formula, respectively. This is the reason why this approximate formula is applicable to a wide energy region. The term including  $B$ , which is proportional to  $\sin \delta$ , is related to T violation [31, 32, 33, 34]. However it is difficult to observe only this term in future long baseline experiments. Therefore there are many attempts to extract the information on the CP phase from the terms including both the coefficients  $A$  and  $B$  [35, 36, 37].

Next let us compare our approximate formula with the exact one. We use the parameters  $\Delta m_{21}^2 = 7.0 \times 10^{-5}$  eV $^2$ ,  $\Delta m_{31}^2 = 2.0 \times 10^{-3}$  eV $^2$ ,  $\sin^2 2\theta_{12} = 0.8$ ,  $\sin^2 2\theta_{23} = 1$ ,  $\sin \theta_{13} = 0.16$ , the oscillation length is  $L = 730$  km  $a = \sqrt{2}G_F N_e$ , where  $G_F$  is the Fermi constant and  $N_e$  is the electron density in matter calculated from the matter density  $\rho = 3$  g/cm $^3$  and the electron fraction  $Y_e = 0.5$ . We plot the coefficients  $A, B$  and  $C$  as a function of the energy within the region  $0.01\text{GeV} \leq E \leq 1\text{GeV}$ . These coefficients calculated from the exact formula are compared with those from the approximate formula in Fig. 1. From this figure we find that the approximate formula almost coincide with the exact formula. The

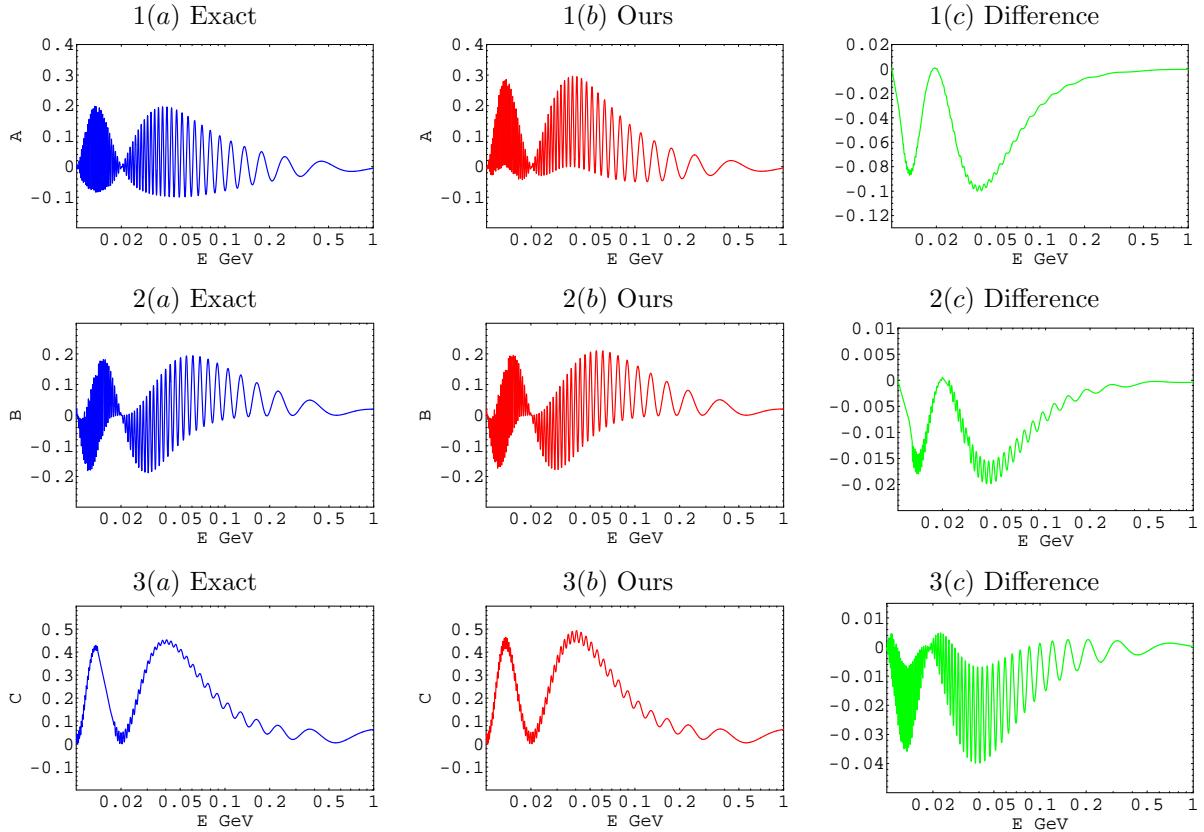


Figure 1: Comparison of our reduction formula with the exact one in the coefficients  $A, B$  and  $C$ , plotted from top to bottom and the exact, approximate formula and the difference from left to right.

error is estimated to be less than 20 % from Fig. 1. The difference between the exact and the approximate formulas is caused by ignoring the higher order terms in the perturbative expansion on  $\sin \theta_{13}$  and  $\alpha$ .

## 4 Derivation of Other Approximate Formulas

In this section, we derive the approximate formulas given by other authors in constant matter from our formula. There are formulas for the low energy [10, 11, 12], the intermediate energy [13, 14, 15] and the high energy regions [16, 17, 18].

### 4.1 Low Energy Formula

At first we derive a low energy formula with large mixing angle  $\theta_{12}$ , which is similar to those in [10, 11, 12]. Under the low energy condition

$$a \ll \Delta_{31}, \quad (76)$$

the following relation

$$\frac{\Delta_{31}^h}{\Delta_{31}} = \frac{\sin 2\theta_{13}}{\sin 2\theta_{13}^h} \simeq 1 \quad (77)$$

is derived by expanding  $\Delta_{31}^h$  and  $\sin 2\theta_{13}^h$  in terms of  $a/\Delta_{31}$ . Namely  $\Delta_{31}^h$  and  $\sin 2\theta_{13}^h$  in matter can be approximated by the quantities in vacuum. Furthermore, if we take the limit  $\theta_{12} \rightarrow \pi/4$

$$\frac{\Delta_{21}^\ell}{\Delta_{21}} = \frac{\sin 2\theta_{12}}{\sin 2\theta_{12}^\ell} \simeq \frac{\sqrt{\Delta_{21}^2 + a^2}}{\Delta_{21}} \quad (78)$$

is obtained. By using the relations (77) and (78), the coefficients  $A$ ,  $B$  and  $C$  for (73), (74) and (75) are reduced to the following expressions

$$A \simeq \frac{\Delta_{21}}{\sqrt{\Delta_{21}^2 + a^2}} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{\Delta_{31} L}{2} \sin \frac{\sqrt{\Delta_{21}^2 + a^2} L}{2} \cos \frac{\Delta_{32} L}{2}, \quad (79)$$

$$B \simeq \frac{\Delta_{21}}{\sqrt{\Delta_{21}^2 + a^2}} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{\Delta_{31} L}{2} \sin \frac{\sqrt{\Delta_{21}^2 + a^2} L}{2} \sin \frac{\Delta_{32} L}{2}, \quad (80)$$

$$C \simeq \frac{\Delta_{21}^2}{\Delta_{21}^2 + a^2} c_{23}^2 \sin^2 2\theta_{12} \sin^2 \frac{\sqrt{\Delta_{21}^2 + a^2} L}{2} + s_{23}^2 \sin^2 2\theta_{13} \sin^2 \frac{\Delta_{31} L}{2}, \quad (81)$$

where the condition derived from the low energy condition (76)

$$\sin \frac{\Delta_{31}^h L}{2} \simeq \sin \frac{\Delta_{31} L}{2} \quad (82)$$

is also used. The applicable region for energy is given by

$$E \ll 15 \text{ GeV} \left( \frac{\Delta m_{31}^2}{10^{-3} \text{ eV}^2} \right) \left( \frac{3 \text{ g/cm}^3}{\rho} \right) \quad (83)$$

from the condition (76). In addition to this condition, the applicable region of (79), (80) and (81) is restricted by

$$L \ll 8000 \text{ km} \left( \frac{E}{\text{GeV}} \right) \left( \frac{10^{-4} \text{ eV}^2}{\Delta m_{21}^2} \right), \quad (84)$$

which comes from the approximation in the oscillation parts of (79), (80) and (81). A similar result can be obtained for the perturbation of  $\sin \theta_{13}$  [19]. They have proposed a low energy formula in arbitrary matter, by using the first order perturbative calculations. Our method has the advantage that the calculation is much simpler. This approximate formula coincides with that in vacuum in the low energy limit, or in other words, this result recovers the vacuum mimicking phenomenon which has been discussed in [12, 38].

## 4.2 Intermediate Energy Formula

At first we derive the intermediate energy formula [13, 14, 15] from our formula. Under the low energy condition

$$a \ll \Delta_{31}, \quad (85)$$

we expand  $\Delta_{31}^h$  and  $\sin 2\theta_{13}^h$  up to first order of  $a/\Delta_{31}$

$$\Delta_{31}^h \simeq \Delta_{31} - 2a \cos 2\theta_{13}, \quad (86)$$

$$\sin 2\theta_{13}^h \simeq \sin 2\theta_{13} \left( 1 + \frac{2a}{\Delta_{31}} \cos 2\theta_{13} \right). \quad (87)$$

Substituting (86) and (87) into (72), we obtain the expression

$$|S_{\tau e}^h|^2 \simeq s_{23}^2 \sin^2 2\theta_{13} \left( 1 + \frac{2a}{\Delta_{31}} \cos 2\theta_{13} \right) \sin^2 \frac{(\Delta_{31} - 2a \cos 2\theta_{13})L}{2} \quad (88)$$

$$\simeq s_{23}^2 \sin^2 2\theta_{13} \left[ \left( 1 + \frac{2a}{\Delta_{31}} \cos 2\theta_{13} \right) \sin^2 \frac{\Delta_{31}L}{2} - aL \cos 2\theta_{13} \sin(\Delta_{31}L) \right], \quad (89)$$

where we also use the approximation

$$aL \cos 2\theta_{13} \ll 1 \quad (90)$$

from the first line to the second line. Furthermore, under the assumption that  $\Delta_{21}L/2$  is small, we can approximate

$$\sin \frac{\Delta_{21}^\ell L}{2} \simeq \frac{\Delta_{21}^\ell L}{2} \simeq \frac{\Delta_{21}L}{2}, \quad (91)$$

and from (73), (74), (75), the approximate formula for  $A$ ,  $B$  and  $C$  is derived as

$$A \simeq \frac{1}{2} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \frac{\Delta_{21}L}{2} \sin(\Delta_{31}L), \quad (92)$$

$$B \simeq \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \frac{\Delta_{21}L}{2} \sin^2 \frac{\Delta_{31}L}{2}, \quad (93)$$

$$C \simeq s_{23}^2 \sin^2 2\theta_{13} \left[ \left( 1 + \frac{2a}{\Delta_{31}} \cos 2\theta_{13} \right) \sin^2 \frac{\Delta_{31}L}{2} - aL \cos 2\theta_{13} \sin(\Delta_{31}L) \right]. \quad (94)$$

One of the conditions for the applicable region of this approximate formula, namely for the upper limit

$$E \ll 15 \text{ GeV} \left( \frac{\Delta m_{31}^2}{10^{-3} \text{ eV}^2} \right) \left( \frac{3 \text{ g/cm}^3}{\rho} \right) \quad (95)$$

is the same as that in the low energy region (76). In addition to this, the conditions due to (90) and (91)

$$L \ll 1700 \text{ km} \left( \frac{3 \text{ g/cm}^3}{\rho} \right), \quad (96)$$

$$E \gg 0.185 \text{ GeV} \left( \frac{\Delta m_{21}^2}{10^{-4} \text{ eV}^2} \right) \left( \frac{L}{730 \text{ km}} \right) \quad (97)$$

should be satisfied. This approximate formula has been derived by using the perturbations of  $\alpha$  and  $a/\Delta m_{31}^2$  [13]. From (95), (96) and (97), the applicable region is rather restricted because of the expansion of the oscillation part. On the other hand, it has the advantage that the contribution of the genuine CP violation can be easily distinguished from that of the fake CP violation.

### 4.3 High Energy Formula

Next, we derive the high energy formulas [16, 17, 18] from our formula. Under the high energy condition

$$a \gg \Delta_{21}, \quad (98)$$

we obtain

$$\frac{\Delta_{21}^\ell}{\Delta_{21}} = \frac{\sin 2\theta_{12}}{\sin 2\theta_{12}^\ell} \simeq \frac{a}{\Delta_{21}} \quad (99)$$

by expanding  $\Delta_{21}^h$  and  $\sin 2\theta_{12}^h$  up to the first order of  $\Delta_{21}/a$ . In addition, using the approximation  $\theta_{13} \rightarrow 0$ , we obtain another relation

$$\frac{\Delta_{31}^h}{\Delta_{31}} = \frac{\sin 2\theta_{13}}{\sin 2\theta_{13}^h} \simeq 1 - \frac{a}{\Delta_{31}}. \quad (100)$$

The concrete expressions of  $A$ ,  $B$  and  $C$  are derived as

$$A \simeq \frac{\Delta_{21}\Delta_{31}}{a(\Delta_{31}-a)} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{aL}{2} \sin \frac{(\Delta_{31}-a)L}{2} \cos \frac{\Delta_{32}L}{2}, \quad (101)$$

$$B \simeq \frac{\Delta_{21}\Delta_{31}}{a(\Delta_{31}-a)} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \frac{aL}{2} \sin \frac{(\Delta_{31}-a)L}{2} \sin \frac{\Delta_{32}L}{2}, \quad (102)$$

$$C \simeq \frac{\Delta_{21}^2}{a^2} c_{23}^2 \sin^2 2\theta_{12} \sin^2 \frac{aL}{2} + \frac{\Delta_{31}^2}{(\Delta_{31}-a)^2} s_{23}^2 \sin^2 2\theta_{13} \sin^2 \frac{(\Delta_{31}-a)L}{2}, \quad (103)$$

by substituting (99) and (100) into (73), (74) and (75), where we also use the approximation

$$\sin \frac{\Delta_{21}^\ell L}{2} \simeq \sin \frac{aL}{2}. \quad (104)$$

The applicable region of this approximate formula is calculated from (98) as

$$E \gg 0.45 \text{ GeV} \left( \frac{\Delta m_{21}^2}{10^{-4} \text{ eV}^2} \right) \left( \frac{3 \text{ g/cm}^3}{\rho} \right). \quad (105)$$

In addition to this, the applicable region is also restricted by

$$L \ll 8000 \text{ km} \left( \frac{E}{\text{GeV}} \right) \left( \frac{10^{-4} \text{ eV}^2}{\Delta m_{21}^2} \right), \quad (106)$$

which is derived from (104). Although these high energy approximate formulas (101), (102) and (103) have been derived at first in [17, 18], their derivation is complicated because of the calculation up to the first order perturbation of  $\alpha$ . Here, we have presented the simple derivation of these formulas by using the new idea of taking only the zeroth order perturbation.

## 5 Summary

In this paper, we study the oscillation probability in matter within the framework of three generations. The results are as follows.

1. We have proposed a simple method to approximate the oscillation probability in arbitrary matter. Our method provide an approximate formula in arbitrary matter *without the usual first order perturbative calculations* of the small parameter  $\Delta m_{21}^2/\Delta m_{31}^2$  or  $\sin \theta_{13}$ .
2. The concrete expressions for our approximate formula in constant matter has been derived to investigate the accuracy of the reduction formula (43)-(46). We have shown that our formula is numerically in good agreement with the exact solution with reasonable accuracy.
3. We have shown that both the low energy [10, 11, 12], the intermediate energy [13, 14, 15] and the high energy [16, 17, 18] approximate formulas in constant matter presented by other authors can be easily derived from our formula. This means that our formula is applicable to a wide energy region.

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